SOME FIXED POINT THEOREMS FOR ISHIKAWA ITERATIONS

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Abstract

This chapter focuses on the convergence of certain Ishikawa type iterations to fixed points of maps satisfying the contractive conditions defined in the earlier chapter. This chapter embibes some fixed point theorems for contractive conditions using Ishikawa iterations established by Albert K. Kalinde and B.E. Rhoades, Kalishankar Tiwary and S.C. Debnath and Rhoades.

Introduction

In 1997, B.E. Rhoades established these following results for Ishikawa Iterations.

Theorem 1 : Let a Banach space X has closed convex subset E and an Ishikawa Iteration scheme satisfy the condition $\underline{\lim}\ \alpha_n > 0$ and $x_n \to a_0$.

Let us consider constants $\alpha, \beta, \gamma, \delta \geq 0, \beta < 1$ in such a way that for all n(sufficiently large), we can write

$$\|Gx_n - Gy_n\| \leq \alpha \|x_n - Gy_n\| + \beta \|x_n - Gx_n\| \quad (1)$$

$$\text{and } \|Ga_0\text{-}Gx_n\| \leq \alpha \|x_n\text{-}a_0\| + \gamma \|x_n\text{-}Gx_n\| + \delta\|a_0\text{-}Gx_n\| + \beta \max\{\|a_0\text{-}Ga_0\| + \|x_n\text{-}Ga_0\|\} \quad (2)$$

Then a_0 is a fixed point of G.

Proof

The equation $x_{n+1} = (1\text{-}\alpha_n)\, x_n + \alpha_n\, Gy_n$ can written as x=Gz, where $x=\{x_n\}$, $z=\{Gy_n\}$ and A is weighted mean matrix generated by $P_0 > 0$, $P_k = \dfrac{\alpha_k}{\prod_{i=1}^{k}(1-\alpha i)}$, k>0.

The condition $\underline{\lim}\ \alpha_n > 0$ in Rhoades [35] proves that A is equivalent to the convergence and hence $\lim x_n = a_0$ which further implies $Gy_n \to a_0$. From Eq. (3.4.1), $\overline{\lim}\, \|Gx_n\text{-}a_0\| \leq \beta\, \overline{\lim}\, \|Gx_n\text{-}a_0\|$ which means $Gx_n \to a_0$. By Eq. (3.4.2), applying limits $n \to \infty$, gives $\|Ga_0 - a_0\| \leq \beta \|Ga_0\text{-}a_0\|$ and $Ga_0 = a_0$

Corollary 2 : Let X be a Banach space and E be a closed convex subset of X. Let G be a self map of E satisfying atleast one of the following conditions for $x, y \in X$:

1 $|x - Gx\| + \|y\text{-}Gy\| \leq a \|x\text{-}y\|$, $1 \leq a < 2$

2 $\|x\text{-}Gx\| + \|y\text{-}Gy\| \leq b[\|x\text{-}Gy\|\ \|y\text{-}Gx\| + \|x\text{-}y\|]$, $½ \leq b < 2/3$

3 $\|x\text{-}Gx\| + \|y\text{-}Gy\| + \|Gx\text{-}Gy\| \leq c\, [\|x\text{-}Gy\|\ \|y\text{-}Gx\|]$ $1 \leq c < 3/2$

4 $\|Gx\text{-}Gy\| \leq k \max\{\|x\text{-}y\|, \|x\text{-}Gx\|, \|y\text{-}Gy\|, ½\, [\|x\text{-}Gy\| + \|y\text{-}Gx\|]$, $0 \leq k < 1$

If Ishikawa iteration scheme, with $\{\alpha_n\}$ bounded away from zero converges to a point a_0, then G converges to fixed point a_0 or a_0 is fixed point of G.

Proof

From Theorem 1, Rhoades derived the following result

$$\|Gx\text{-}Gy\| \leq k \max\{c\|x\text{-}y\|, \|x\text{-}Gx\| + \|y\text{-}Gy\|, \|x\text{-}Gy\| + \|y\text{-}Ga\|\} \quad (3)$$

Where G is self map of E which is a closed convex subset of a Banach space X, c, k ≥0, k < 1. which can be written in the form

$$\|Gx_n\text{-}Gy_n\| \leq \left\{a+1, \frac{2(1+b)}{1-b}, \left(\frac{1+2c}{2-c}\right), \frac{k}{1-k}\right\} \|Gx_n\text{-}Gy_n\| \quad (4)$$

Using Triangle inequality, we have

$$\|Gx_n\text{-}Ga_0\| - \|x_n\text{-}Ga_0\| \leq \|Gx_n - x_n\| \text{ and } \|Gx_n\text{-}Ga_0\| - \|a_0\text{-}Gx_n\| \leq \|a_0\text{-}Ga_0\|$$

If x_n and a_0 satisfy condition (1) of this theorem, then

$$\|Gx_n\text{-}Ga_0\| \leq \frac{a}{2}[\|a_0\text{-}x_n\| + \|x_n\text{-}Ga_0\| + \|a_0\text{-}Gx_n\|]$$

If x_n and a_0 satisfy condition (2) of this theorem, we get

$$\|Gx_n-Ga_0\| \le \frac{1+b}{2}[\|xn-Ga_0\| + \| a_0-Gx_n\|] + \frac{b}{2}\|a_0-x_n\|$$

If x_n and a_0 satisfy condition (3) of this theorem, we get

$$\|Gx_n - Ga_0\| \le \frac{c+1}{3}[\|x_n-Gx_n\| + \| a_0 - Gx_n \|]$$

If x_n and a_0 satisfy condition (4) of this theorem, we get

$$\|Gx_n - Ga_0 \| \le k \max\left\{\|x_n-a_0\|, \|x_n-Gx_n\|, a_0-Ga_0\|, \frac{1}{2}[\|x_n-Ga_0\|+\|a_0-Gx_n\|]\right\}$$

For all these four cases,

$$\|Gx_n-Ga_0\|\le k \max\left\{\frac{a}{2},\frac{b}{2},k\right\}\|x_n-a_0\| + \max\left\{\frac{c+1}{3},k\right\}\|x_n-Gx_n\| +$$

$$\max\left\{\frac{a}{2},\frac{1+b}{2},\frac{1+c}{3},\frac{k}{2}\right\}\|a_0-Gx_n\|+ \max\left\{\frac{a}{2},\frac{1+b}{2},k\right\}\max\{\|a_0-Ga_0\|, \|x_n-Ga_0\|\}$$

and equation (2) is satisfied.

Corollary : Let a Banach space X has a closed convex subset E and G is a self map of E satisfying atleast one of these following conditions on x, y in X.

1 $b_0(Gx-Gy)\le k \max\{b_0(x-y), b_0(x-Gx), b_0(y-Gy), b_0(x-Gy), b_0(y-Gx)) \; 0\le k < 1$ (5)

2 $b_0(Gx-Gy)+b_0(x-Gx)+b_0(y-Gy)\le c[b_0(x-Gy)+b(y-Gx) \; 0\le c<2$ (6)

The sequence $\{x_n\}$ of Ishikawa iterates converge to a point a_0, where $\lim\alpha_n>0$ and $x_0\in E$. Then a_0 is a fixed point of G.

Proof

If x_n, y_n satisfy the equation (5) then,

$$b_0 (Gx_n-Gy_n)\le k[2b_0(x_n-Gy_n) + b_0(Gx_n-Gy_n)]$$

If x_n, y_n satisfy (6), then

$$3 b_0(Gx_n - Gy_n) \le (1+c) [b_0(Gy_n-Gx_n)+ 2 b_0(x_n-Gx_n)]$$

In either case

$$b_0 (Gx_n-Gy_n) \le \max\left\{\frac{2k}{1-k},\frac{2(1+c)}{2-c}\right\} b_0(x_n-Gy_n) \text{ and}$$

Eq. (1) is satisfied

If x_n, a_0 satisfy (5), then

$$b_0(Gx_n-Ga_0)\le k \max\{b_0(x_n-a_0),b_0(x_n-Gx_n), b_0(a_0-Ga_0),b_0(x_n-Ga_0)+b_0(a_0-Gx_n)\}$$

$$\le k\, b_0(x_n-a_0)+k\, b_0(x_n-Gx_n)+ k\, b_0(a_0-Gx_n)+k \max\{b_0(a_0-Ga_0), b_0(x_n-Ga_0)\}$$

If x_n, a_0 satisfy (3.4.6), then we get

$$3 b_0 (Gx_n - Ga_0) \le (1+c) [b_0(x_n-Ga_0)+b_0(a_0-Gx)]$$

In either case,

$$b_0(Gx_n-Ga_0)\le k\, b_0(x_n-a_0)+k\, b_0(x_n-Gx_n) + \max\left\{k,\frac{1+c}{3}\right\} b_0 (a_0-Gx_n)$$

$$+ \max\left\{k,\frac{1+c}{3}\right\}\max \{b_0(a_0-Ga_0), b_0(x_n-Ga_0)\},$$

and Eq. (2) is satisfied.

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