International Advance Journal of Engineering, Science and Management (IAJESM)
ISSN -2393-8048, July-December 2022, Submitted in July 2022, iajesm2014@gmail.com

A Class of Univalent Analytic Functions Involving the Ruscheweyh Derivatives Operator and Hadamard Product

Pooja Lakhera, Research Scholar, Department of Mathematics, SunRise University, Alwar.
Dr. Rajeev Kumar, Associate Professor, Department of Mathematics, SunRise University, Alwar.

This section is devoted to univalent mappings defined hadamard product involving a Ruscheweyh Derivatives Op_{tor} .

Let $\mathcal{A}$ stand for the class of mapping

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1.1}$$

whichever regular and one to one in the $\mathcal{U}_D$ $\mathcal{U} = \{z/|z| < 1\}$

For $g(z)=z+\sum_{k=2}^{\infty}b_k z^k$ (1.2)

the Hd_{pro} of f(z) & g(z) be

$$(f*g)=z+\sum_{k=2}^{\infty}a_k b_k z^k,\ z\in\mathcal{U} \tag{1.3}$$

Let D^mf(z) indicate the m[th] order derivative

The Ruschweyh derivative is definite go with D^m:S→ S like wise

$$D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), \qquad m> -1$$

$$= \frac{z\left(z^{m-1}f(z)\right)^m}{m!} \qquad m\in N_0 = 0,1,2$$

$$=z+\sum_{m=2}^{\infty}a_k B(m,k)z^k \tag{1.4}$$

wherever B(m,k) = $\binom{m+k-1}{m}$

note that $\frac{z}{(1-z)^{m+1}}=z+\sum_{m=2}^{\infty}a_k B(m,k)z^k$ where m> -1

as well as $\quad D^0 f(z) = f(z), D' f(z) = zf'(z)$

Now f∈ $\mathcal{A}$ & satisfy

$$\left|\frac{z\frac{D^{m+1}f(z)}{D^m f(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{D^{m+1}f(z)}{D^m f(z)}}\right|<\delta \tag{1.5}$$

For 0≤ $\alpha < 1$, $\quad$ 0< $\delta \le 1$, $\quad$ 0≤ $\lambda < \mu \le 1$

The study here is inspired by S. Khairnar & M.More [14]

1.1 Coefficient Inequality

$\mathcal{Theorem}$ **1:**Let f∈ S , Then f∈ $H(\alpha,\delta,\mu,\lambda)$ iff.

$$\sum_{k=2}^{\infty}\left[(k-1)+\delta\left((k-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,k)a_k\le\delta(1-\alpha)(\mu-\lambda) \tag{1.6}$$

Proof **:** Suppose f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ by (1.4)

$$\left|\frac{z\frac{D^{m+1}f(z)}{D^m f(z)}-1}{(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{D^{m+1}f(z)}{D^m f(z)}}\right|<\delta$$

$$D^m f(z)=z+\sum_{m=2}^{\infty}a_k B(m,k)z^k$$

$$D^{m+1}f(z)=1+\sum_{m=2}^{\infty}ka_k B(m,k)z^{k-1}$$

$$zD^{(m+1)}f(z)-D^m f(z)=z+\sum_{m=2}^{\infty}ka_k B(m,k)z^k-z-\sum_{m=2}^{\infty}a_k B(m,k)z^k$$

$$=\sum_{k=2}^{\infty}(k-1)a_k B(m,k)z^k$$

Now

$$(1-\alpha)\mu+\alpha\lambda-\lambda z\frac{D^{m+1}(z)}{D^m(z)}=[(1-\alpha)\mu+\alpha\lambda]D^m(z)-\lambda zD^{m+1}(z)$$

$$=[(1-\alpha)\mu+\alpha\lambda]\left[z+\sum_{m=2}^{\infty}a_k\,B(m,k)z^k\right]-\lambda z\left[1+\sum_{m=2}^{\infty}ka_k\,B(m,k)z^{k-1}\right]$$

$$=[(1-\alpha)\mu+\alpha\lambda-\lambda]z+\sum_{m=2}^{\infty}[(1-\alpha)\mu+\alpha\lambda-\lambda k]\,a_k B(m,k)z^k$$

$$=\left[(1-\alpha)\mu-(1-\alpha)\lambda\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k$$

$$=\left[(1-\alpha)(\mu-\lambda)\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k$$

$$\left|\frac{\sum_{m=2}^{\infty}(k-1)a_k B(m,k)z^k}{\left[(1-\alpha)(\mu-\lambda)\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k}\right|<\delta \qquad (1.7)$$

We be familiar with $|Re(z)|<|z|$

$$\mathrm{Re}\left|\frac{\sum_{m=2}^{\infty}(k-1)a_k B(m,k)z^k}{\left[(1-\alpha)(\mu-\lambda)\right]z+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)z^k}\right|<\delta$$

We choose values of greater than expression & allowing z→ 1 throughout the $\mathcal{R}e_{al}$ value we acquire

$$\sum_{m=2}^{\infty}(k-1)a_k B(m,k) \le \delta\left[(1-\alpha)(\mu-\lambda)+\sum_{m=2}^{\infty}\left[(1-\alpha)\mu-(k-\alpha)\lambda\right]a_k B(m,k)\right]$$

$$\therefore \sum_{m=2}^{\infty}[(k-1)+\delta[(k-\alpha)\lambda-(1-\alpha)\mu]a_k\, B(m,k) \le \delta(1-\alpha)(\mu-\lambda)$$

Conversely

Suppose that

$$\sum_{m=2}^{\infty}[(k-1)+\delta[(k-\alpha)\lambda-(1-\alpha)\mu]a_k\, B(m,k) \le \delta(1-\alpha)(\mu-\lambda)$$

Without help enclose

$|zD^{m+1}f(z)-D^m f(z)|-\delta|(1-\alpha)\mu+\alpha\lambda-\lambda zD^{m+1}f(z)|<0$

With the position

$$\left|\sum_{m=2}^{\infty}(k-1)a_k B(m,k)z^k\right|-\delta\left|(1-\alpha)(\mu-\lambda)z+\sum_{m=2}^{\infty}(1-\alpha)\mu(k-\lambda)a_k B(m,k)z^k\right|$$

For $|z|<r<1$ the condition is bounded above

$$\sum_{m=2}^{\infty}(k-1)a_k\, B(m,k)r^k$$

$$-\delta\left[(1-\alpha)(\mu-\lambda)z+\sum_{m=2}^{\infty}(1-\alpha)\mu(k-\lambda)a_k B(m,k)z^k\right]<0$$

$$\sum_{m=2}^{\infty}\left[(k-1)+\delta\left[(k-\alpha)\lambda-(1-\alpha)\mu\right]a_k B(m,k)\le\delta(1-\alpha)(\mu-\lambda)\right.$$

∴ f(z)∈ $H(\alpha,\delta,\mu,\lambda)$

Corollary **1:** If f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ then

$$a_k \le \frac{\delta(1-\alpha)(\mu-\lambda)}{\left[(k-1)+\delta\left((k-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,k)} \qquad (1.8)$$

and equality holds for

$$f(z)=z+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[(k-1)+\delta\left((k-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,k)}z^k$$

4.2.2 Growth And Distortion Theorem

Theorem **2 :**Whenever the mapping f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ then

$$|z|-\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2\le|f(z)|$$
$$\le|z|+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2 \qquad (1.9)$$

With the equality for

$$f(z)=z+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}z^2 \tag{1.10}$$

Proof **:** from theorem (1) f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ iff

$$\sum_{m=2}^{\infty}[(k-1)+\delta((k-\alpha)\lambda-(1-\alpha)\mu)]B(m,k)a_k \leq \delta(1-\alpha)(\mu-\lambda)$$

Now

$$|f(z)| \leq |z|+a_k|z|^k$$

$$|f(z)| \leq |z|+|z|^2\sum_{k=2}^{\infty}|a_k|$$

$$|f(z)| \leq |z|+|z|^2\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}$$

$$|f(z)| \leq |z|+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2 \tag{1.11}$$

Simillary

$$|f(z)| \geq |z|-\sum_{k=2}^{\infty}a_k|z|^k$$

$$|f(z)| \geq |z|-|z|^2\sum_{k=2}^{\infty}|a_k|$$

$$|f(z)| \geq |z|-|z|^2\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}$$

$$|f(z)| \geq |z|-\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2 \tag{1.12}$$

By (1.11) and (1.12) we obtain

$$|z|-\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2 \leq |f(z)|$$

$$\leq |z|+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}|z|^2$$

with the equality for

$$f(z)=z+\frac{\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta\left((2-\alpha)\lambda-(1-\alpha)\mu\right)\right]B(m,2)}z^2$$

which complete the proof.

Theorem **3:** If the mapping f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ then

$$1-\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}|z|\le|f'(z)|\le 1+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}|z| \quad (1.13)$$

The equality hold for

$$f(z) = z+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}z^2 \quad (1.14)$$

Proof **:** From theorem (1) f(z)∈ $H(\alpha,\delta,\mu,\lambda)$ if and only if

$$\sum_{m=2}^{\infty}\left[(k-1)+\delta((k-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,k)a_k\le\delta(1-\alpha)(\mu-\lambda)$$

Now

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k$$

$$f'(z)=1+\sum_{k=2}^{\infty}k a_k z^{k-1}$$

$$|f'(z)|\le 1+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}|z|$$

Simillarly

$$|f'(z)|\ge 1-\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}|z|$$

$$\therefore\ 1-\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}|z|\le|f'(z)|\le 1+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}|z|$$

The equality hold for

$$f(z) = z+\frac{2\delta(1-\alpha)(\mu-\lambda)}{\left[1+\delta((2-\alpha)\lambda-(1-\alpha)\mu)\right]B(m,2)}z^2$$

which complete the proof.

REFERENCES

1. H.M. Srivastava and S. Owa(1986),A new class of analytic functions with negative coefficients, Comment.Math.Univ.St.Paul,35,175-188.
2. Huseyin Irmak and R.K. Raina (2004), Some applications of generalized fractional calculus operators to a novel class of analytic functions with negative coefficients, Taiwanese Journal of Mathematics, Vol.8, No.3, 443-452.
3. Jadhav P.G. and Khairnar S.M.(2013), Certain family of meromorphic and multivalent functions defined by linear operator in the unit disc, International J. of Math. Sci. and Engg. Appls (IJMSEA),7,15-33.
4. Jadhav P.G. and Khairnar S.M.(2012),Certain subclasses of analytic and multivalent functions in the unit disc, International J.of Math. Sci, and Engg. Appls. (IJMSEA),Vol.6,Nov.,175-192.

5. Jung, I. B., Kim, Y. C. and Srivastava, H. M. (1993), The hardly space of analytic functions associated with certain one-parameter families of integral transform, J. Math. Anal. Appl., 179 , 138-147.(8.3)
6. Lee S.K., Khairnar S.M. and Rajas S.M.(2011),On a certain class of p-valent uniformly convex functions using differential operator, Korean J.Math.,19,1-16.(3.2.4)
7. M. Darus (2003), Some subclass of analytic functions, Journal of Math. and Comp. Sci. (Math Ser), 16 , 121-126 .
8. Mohamed Kamal Aouf, Daniel Breaz, Nicoleta Breaz (2011),Inequalities Involving Noor Integral Operator, Theory and Applications of Mathematics & Computer Science 1 ,63–67.
9. More, M. and Khairnar, S.M. (2007),Properties of a Subclass of Meromorphic Univalent Functions With Positive Coefficients, International J. of Math. Sci. & Engg. Appls. 1, 101-113. (4.2.1)
10. S. Khairnar and Meena More (2007),Properties of a subclass of meromorphic univalent functions with positive coefficients, International J. of Math. Sci. and Engg. Appls., 1, 101-113.
11. S. Khairnar and Meena More (2007),Subclass of analytic and univalent functions in the unit disk, Scientia Magna, 3, 1-8.(8.2)
12. S. Khairnar and Meena More (2007),Convolution properties of univalent functions with missing second coefficient of alternating type, International. J. of Maths. Sci. and Engg. Appls.Vol. I, No.1, 101-113.